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Kinetic Theory Analysis of Diffusion of Discontinuity Plane of Flow Velocity

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PROBLEMS in which state variables (e.g., density, velocity, etc.) change considerably in a short range (comparable to mean free path) in a gas are of main interest in rarefied gasdynamics as well as those[1,3] in which the conditions of a solid boundary in a gas change in a short time (comparable to mean collision period of gas molecules). As a simple and fundamental example of the former phenomena, we here try to investigate how a plane of initial discontinuity in gas velocity diffuses on the basis of the Boltzmann equation with *B-G-K* model.[4]

At time $t = 0$,‡ the gas is assumed to have uniform velocity U in x direction in the region $y > 0$ and $-U$ in $y < 0$, uniform density ρ_0 and uniform temperature T_0, and to be in equilibrium in respective regions. The plane at $y = 0$ of initial discontinuity in gas velocity is released to diffuse for $t > 0$. We further assume that the initial velocity U is much less than the velocity of sound in the gas so that the fundamental equations as well as the initial and boundary conditions may be linearized. Then, we can show that the density and temperature of the gas remain constant for $t > 0$. More generally, we can treat the velocity in x direction independently of the density, temperature, and velocity in y direction in the problem where the initial density and temperature are also not uniform.§ Accordingly, our result given below may represent the field of velocity in x direction in this more general problem.

On the basis of the *B-G-K* model, the linearized kinetic equations become

$$\left.\begin{aligned} \partial\phi\,\partial t + v\,\partial\phi\,\partial y &= \lambda(-\phi + 2huq_x) \\ q_x &= \iiint_{-\infty}^{\infty} u\phi F d\mathbf{v} \\ F = (h/\pi)^{3/2}\exp\{-h\,(u^2 + v^2 + w^2)\} &\qquad h = m/2kT_0 \end{aligned}\right\} \quad (1)$$

where $(\rho_0/m)F(1 + \phi)$ is the velocity distribution function, q_x the x component of the gas velocity, m the mass of a molecule, $\mathbf{v} = (u, v, w)$ the molecular velocity, k the Boltzmann constant, and λ a constant (collision frequency) related to the mean free path l as follows: $l = 2/\{(\pi h)^{1/2}\lambda\}$. The initial condition is

$$\left.\begin{aligned} \phi &= 2huU \qquad (y > 0) \\ &= -2huU \qquad (y < 0) \end{aligned}\right\} \quad \text{at } t = 0 \qquad (2)$$

whereas the boundary conditions are

$$\begin{aligned} \phi &= 2huU \qquad (v < 0) \qquad \text{as } y \to \infty \\ &= -2huU \qquad (v > 0) \qquad \text{as } y \to -\infty \end{aligned} \qquad (3)$$

ϕ is continuous at $y = 0$. From (1–3) we obtain, after some reduction, the following integral equation for q_x:

$$\frac{\bar{q}_x}{U} = \frac{\pm 1}{s + \lambda}\left\{1 - \frac{2}{\pi^{1/2}} J_0[h^{1/2}(s + \lambda)|y|]\right\} + \lambda\left(\frac{h}{\pi}\right)^{1/2}\int_{-\infty}^{\infty}\frac{\bar{q}_x}{U} J_{-1}[h^{1/2}(s + \lambda)|y - y_0|]dy_0 \qquad (4)$$

where the bar over a letter indicates the Laplace transform ($t \to s$), the upper sign holds for $y > 0$ and the lower for $y < 0$, and the functions J_n's are defined by the integral[5]

$$J_n(\xi) = \int_0^{\infty} \zeta^n \exp\left[-\zeta^2 - \frac{\xi}{\zeta}\right] d\zeta$$

For short times ($\lambda t \ll 1$), the state of affairs is expected to be close to the free molecular flow ($\lambda = 0$). For free molecular flow, the right-hand side of Eq. (4) degenerates into its first term, and we easily obtain the solution

$$\frac{q_x}{U} = \frac{2}{\pi^{1/2}}\int_0^{h^{1/2}y/t}\exp(-\zeta^2)d\zeta \qquad (5)$$

The solution for short times for our problem may be obtained by adding a perturbation to this result. Thus, we obtain

$$\frac{q_x}{U} \cong \frac{2h^{1/2}y}{\pi^{1/2}t}\,[1 + \lambda t\{2^{1/2}\log(2^{1/2} + 1) - 1\}] \qquad \left|\frac{h^{1/2}y}{t}\right| \ll 1 \qquad (6a)$$

$$\frac{q_x}{U} \cong 1 - \frac{t}{\pi^{1/2}h^{1/2}y}\left[1 - \frac{1}{2}\left(\frac{t}{h^{1/2}y}\right)^2 - \lambda t\left\{1 - \frac{3}{2}\left(\frac{t}{h^{1/2}y}\right)^2\right\}\right]\exp\left\{-\left(\frac{h^{1/2}y}{t}\right)^2\right\} \qquad \left|\frac{h^{1/2}y}{t}\right| \gg 1 \qquad (6b)$$

The effect of molecular collision at the initial stage appears in the following way. Molecules coming from the region $y > 0$ have lost some of their average velocity in x direction by collision with molecules that have emerged from $y < 0$, whereas molecules from $y < 0$ have obtained the average velocity by collision with molecules from $y > 0$. For $y > 0$, the latter effect dominates the former. Thus, the flow is less decelerated than the free molecular flow there. In other words, the diffusion (or mixing) is slowed down by molecular collisions. It is also noted that the result (6a) does not contain any nonanalytic term such as $(h^{1/2}y/t)\log(h^{1/2}y/t)$ in contrast to the case in which a solid boundary exists as in Rayleigh flow.[1,2] Figure 1 shows the distribution of gas velocity against $h^{1/2}y/t$ for $\lambda t = 0.4$ (the result for free molecular flow is also shown for comparison).

For $\lambda t \gg 1$, the field may be described by the classical result based on the Navier-Stokes equation. That is, the classical solution with kinematic viscosity $\nu = 1/2h\lambda$ (corresponding to *B-G-K* model)

$$\frac{q_x}{U} = \frac{2}{\pi^{1/2}}\int_0^{h^{1/2}\lambda y/(2\lambda t)^{1/2}}\exp(-\xi^2)\,d\xi \qquad (7)$$

may be seen to satisfy the integral equation (4) by direct substitution, if we assume t^{-1} and $h^{1/2}y/t$ small¶ and neglect these

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‡ t is time and x, y, z is the Cartesian coordinate system.

§ The proof may be given in quite a similar way as in Ref. 2. Nonuniformity must be small to assure the linearization.

¶ Further study is required to clear the detailed behavior for $y \gtrsim h^{-1/2}t$. The region $y \sim h^{-1/2}t$ is interesting especially in the case where pressure pulse (sound wave) travels into the gas as in Ref. 3, since the ridge of the sound pulse is advancing there.

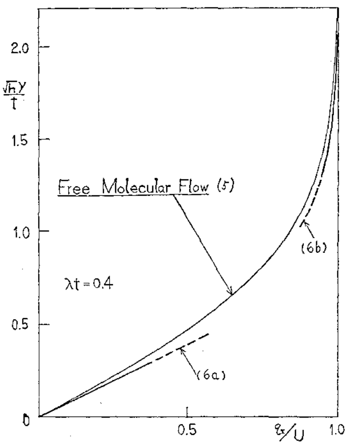

Fig. 1 The distribution of the velocity in the gas ($\lambda t = 0.4$).

and higher order terms. (Note that $h\lambda y^2/t$ may not necessarily be small.) The detailed analysis is similar as in Ref. 2. The preceding result may be compared with the result in Rayleigh flow[1,2] where a solid boundary plays a fundamental role, and a correction proportional to $t^{-1/2}$ to the classical result appears representing a kind of boundary layer with thickness of the order of mean free path. Roughly speaking, for $\lambda t \gg 1$, in the region of viscous diffusion $[y \lesssim (2t/h\lambda)^{1/2}]$, which is much smaller than that of (free) molecular diffusion ($y \lesssim h^{1/2}t$), mixing and collision of molecules occur adequately to assure the validity of the Navier-Stokes equation except near a solid boundary that does not exist in the present problem. It may also be seen from (6) and (7)** that the effective speed of diffusion is slowed down from $O(h^{-1/2})$ to $O\{(2h\lambda t)^{-1/2}\}$ as time progresses.

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** Note that the solution (6) depends on $h^{1/2}y/t$ except for a small perturbation and that (7) is a function of $\{h^{1/2}\lambda y/(2\lambda t)^{1/2}\}$.

Stability Analysis of a Simplified Flexible Vehicle via Lyapunov's Direct Method

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RECENTLY, it was shown that sufficient conditions for asymptotic stability of equilibrium of certain classes of aeroelastic systems with distributed aerodynamic load can be derived via Lyapunov's direct method.[1] This approach has the advantage over the conventional ones in the respect that it allows one to deal directly with the systems' partial differential equations without resorting to any approximations. Moreover, it is potentially applicable to the stability analysis of nonlinear aeroelastic systems. In this note, similar conditions for a simplified aerodynamic vehicle with flexible tail will be derived using the same approach. This aeroelastic system differs from those considered previously in the sense that the aerodynamic load can be approximated by a concentrated force.

Figure 1 shows the tail portion of a flexible vehicle. For the present analysis, it is assumed that the tail motion corresponds approximately to that of a nonuniform cantilever beam in plane bending. For this system, the dimensionless equation of the perturbed motion about its equilibrium state can be given by

$$m(x)v_0^2l^2\frac{\partial^2 w(t,x)}{\partial t^2} + v_0 l^3 k_d(t,x)\frac{\partial w(t,x)}{\partial t} = -\frac{\partial^2}{\partial x^2}EI(x)\frac{\partial^2 w(t,x)}{\partial x^2} \quad (1)$$

where both the beam deflection w and the spatial coordinate x have been normalized with respect to the beam length l; t is the dimensionless time normalized with respect to the quantity l/v_0, where v_0 is the freestream velocity of air; m, k_d, and EI are linear mass density, distributed damping coefficient, and bending rigidity, respectively.

Assuming that the aerodynamic load on the tail lifting surface can be approximated by that of a thin flat plate in an incompressible flow,[3] and the incremental aerodynamic moment due to tail motion is negligible, the boundary conditions have the form

$$w(t,0) = 0 \qquad [\partial w(t,x)/\partial x]|_{x=0} = 0 \quad (2)$$

$$EI(x)\frac{\partial^2 w(t,x)}{\partial x^2}\bigg|_{x=1} = 0 \qquad \frac{\partial}{\partial x}EI(x)\frac{\partial^2 w(t,x)}{\partial x^2}\bigg|_{x=1} = 2\pi\rho_a v_0^2 l^2 ab\left[\frac{\partial w(t,x)}{\partial t} + \frac{\partial w(t,x)}{\partial x}\right]\bigg|_{x=1} \quad (3)$$

where ρ_a is the mass density of the undisturbed air, and a and b are the length and width of the tail lifting surfaces, respectively. The state S_t of this system at any time t can be specified by the functions $w(t, x)$ and $\partial w(t, x)/\partial t$ defined for all $x\epsilon[0, 1]$.

The problem is to derive sufficient conditions for the asymptotic stability of equilibrium of the flexible tail in the sense of Lyapunov[1,2] with respect to a norm defined by

$$||S_t|| = \left\{\int_0^1\left[\left(\frac{\partial w}{\partial t}\right)^2 + \sum_{n=0}^{2}\left(\frac{\partial^n w}{\partial x^n}\right)^2\right]dx\right\}^{1/2} \quad (4)\dagger$$

Note that although the system is linear, the associated boundary-value problem is nonselfadjoint. The determination of conditions for asymptotic stability in terms of the system parameters is not a trivial task.

To apply Lyapunov's direct method to this problem, consider the following functional:

$$V = \frac{1}{2}\int_0^1\left[m(x)v_0^2l^2\left(\frac{\partial w}{\partial t}\right)^2 + 2c_0v_0l^2m(x)\frac{\partial w}{\partial t}\frac{\partial w}{\partial x} + 2\pi\rho_a v_0^2 l^2 ab\left(\frac{\partial w}{\partial x}\right)^2 + EI(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right]dx \quad (5)$$

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† In essence, this norm establishes a measure of the closeness of the system state S_t from the equilibrium null state in terms of the deflection, slope, curvature, and velocity of the beam. Note that for systems having infinite degrees-of-freedom, stability with respect to one norm generally does not imply stability with respect to another norm. The choice of the norm should be based on a careful scrutiny of the physical properties of the particular system under consideration. A detailed discussion of the physical meaning of stability in the sense of Lyapunov is given in Ref. 1.